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LETTER TO THE EDITOR

Inconsistencies in the description of a quantum system with a finite number of bound states by a compact dynamical group

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Abstract
In a recent paper (Aldaya V and Guerrero J 2005 *J. Phys. A: Math. Gen.* **38** 6939) it is shown that the bound states of the Modified Pöschl–Teller potential should be described by the non-compact dynamical group $SU(1,1)$ instead of the usual compact group $SU(2)$. Here we prove that $SU(2)$ cannot be the dynamical group for a potential with a finite number of bound states on the basis of the Modified Pöschl–Teller potential and the Morse potential. This suggests that a quantum system with a continuum spectrum and a finite number of bound states should be described, both in the continuum and the discrete spectrum, by a non-compact dynamical group instead of a compact one.

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1. Introduction

Recently, the authors have shown [1] that the bound states of the Modified Pöschl–Teller (MPT) potential should be described by the dynamical group $SU(1,1) \approx SL(2,\mathbb{R})$ instead of $SU(2)$, as is customarily presented in the literature [4, 5] (see also [6] and references therein for a recent account). The key point has been a deep analysis of the unitarity of the representation under which the eigenstates of the Hamiltonian transform. In fact, unitary irreducible representations of $SU(2)$ require $2j+1$ normalizable states, with j integer or half-integer, but the Hilbert space of (normalizable) bound states for the MPT potential is spanned by only $[j]+1$ states, where $[j]$ stands for the greatest integer strictly smaller than j, and j is related to the potential parameters. That is, not only there are fewer states in the true representation space than in the case of $SU(2)$, but also the parameter characterizing the representations can take any positive value.

These features cannot be accomplished by the representations of a compact group as $SU(2)$, since it is simply connected (therefore we cannot resort to the universal covering group to obtain more allowed values of j), and the only irreducible representations correspond to integer and half-integer j, and these are unitary. Furthermore, there is no well-defined action of $SU(2)$ on a one-dimensional space; it only acts on the sphere S^2, with the spherical harmonics spanning the Hilbert space, or the complex plane (via stereographic projection from the Riemann sphere), with a carrier Hilbert space of holomorphic functions.

In spite of this, in the literature it is commonly stated that $SU(2)$ is the dynamical group for the bound states of the MPT potential, and that a similar situation occurs for other one-dimensional potentials like the Morse potential [5, 7] (see also [8] and references therein). In general, it is assumed that the dynamical group associated with a finite number of bound states must be compact [5]. Also, the explicit construction of the MPT and Morse energy eigenstates in terms of $SU(2)$ wavefunctions has been done [5]. According to the previous discussion, and as was proved in [1], the dynamical group for the bound states of the MPT potential is non-compact (there, it was also suggested that a similar situation occurs for the Morse potential). Therefore, there exists a clear contradiction between our results and those in the literature.

In this paper, we show that the construction on $SU(2)$ grounds is not correct and fails due to the lack of unitarity. The key point is that even though the MPT and Morse eigenfunctions can be described in terms of (a piece of) $SU(2)$ wavefunctions (associated Legendre functions for the MPT potential and associated Laguerre polynomials for the Morse potential, respectively) by means of appropriate changes of variables, the integration measures obtained through these changes do not coincide with the invariant integration measures for $SU(2)$ in the respective spaces. As a consequence, not all states are normalizable with respect to the new integration measure, so that only $[j]+1$ out of the $2j+1$ states are normalizable. Accordingly, these states cannot span an irreducible representation of $SU(2)$. In [1], we proved that they indeed build a new kind of 'unitary', non-local realization of a finite-dimensional, non-unitary, irreducible representation of $SL(2,\mathbb{R})$ for the case of the MPT potential, and it was conjectured that this is also the case of the Morse potential.

It is interesting to note that the claim that the dynamical group for the bound states of the Morse potential is $SU(2)$ is based on [9], where the algebra satisfied by the ladder operators corresponding to an F-type factorization (see [10]) of the Morse Hamiltonian derived in [11], after the introduction of an extra variable, in order to avoid the appearance of the quantum number n in the factorization, is done. But, in [9], it is recognized that although the algebra has as commutation relations those of $SO(3)$, the ladder operators are not adjoint to each other. This clearly indicates that $SO(3)$ (or $SU(2)$) cannot be the dynamical group for the bound states of the Morse potential.

Our statement that $SU(1,1)$ is the dynamical group for the bound states of the MPT potential, realizing a non-unitary, finite-dimensional representation of $SU(1,1)$, differs from (and does not contradict) others in the literature. In fact, in [5, 12], the group $SU(1,1)$ appears as the dynamical group for the scattering states of the MPT potential, realized through the continuos series of representations. Also, in [9, 12], the same group is shown to be the potential group for the bound states of the MPT potential, in the sense that the ladder operators in a certain realization of the discrete series representations of $SU(1,1)$ shift the potential depth, keeping the energy fixed (see also [13] for a description in terms of the coupling of two $SU(1,1)$ representations in the framework of a larger potential group $Sp(2N,\mathbb{R})$). Finally, in the case of the trigonometric Pöschl–Teller potential, having an infinite number of bound states and no scattering states, $SU(1,1)$ plays the role of the dynamical group realized in the discrete series representations (see [14] for a recent account).

The paper is organized as follows. In section 2, we review the representations of $SU(2)$ on the sphere in terms of spherical harmonics, which factorize as the product of a phase and an associated Legendre function, and it is shown that these representations cannot be expressed in terms of just associated Legendre functions. In section 3, the representations of $SU(2)$ are constructed on the plane in terms of two boson operators (Schwinger representation), where the wavefunctions are expressed as the product of a phase and a radial part containing an associated Laguerre polynomial, and we prove that these representations cannot be expressed in terms of just the radial part. In sections 4 and 5, these results are applied to show that $SU(2)$ cannot be the dynamical group for the bound states of the MPT nor the Morse potential, respectively.

All formulae concerning properties of the associated Legendre functions and the associated Laguerre polynomials appearing in this letter are standard and can be seen, for instance, in [15, 16].

2. Representations of $SU(2)$ on the sphere: associated Legendre functions

The most familiar realization of the representations of $SU(2)$ takes place on the sphere S^2 in terms of spherical harmonics (for integer values of spin). In this case, the sphere lays in the physical space, i.e. it is obtained by changing from Cartesian coordinates (x, y, z) to spherical ones (r, θ, ϕ), $r > 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, and taking $r = 1$. The Casimir and the third component of the angular momentum operators $\vec{J} = -\mathrm{i}\vec{r} \times \vec{\nabla}$ are written as:

$$J^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \qquad J_3 = -\mathrm{i}\frac{\partial}{\partial\phi}. \tag{1}$$

A given representation is characterized by the index j, and a basis $\left\{Y_j^m(\theta, \phi)\right\}_{m=-j}^{j}$ for the Hilbert space is determined by the eigenvalue equations $J^2 Y_j^m = j(j+1)Y_j^m$ and $J_3 Y_j^m = m Y_j^m$, whose solutions are the well-known spherical harmonics. They are written as

$$Y_j^m(\theta, \phi) = \sqrt{\frac{(2j+1)}{4\pi}\frac{(j-m)!}{(j+m)!}}\,P_j^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \qquad m = -j, -j+1, \ldots, j-1, j, \tag{2}$$

where $P_j^m(\cos\theta)$ are the associated Legendre functions[4] satisfying the differential equation $(x = \cos\theta)$:

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + j(j+1) - \frac{m^2}{1-x^2}\right]P_j^m(x) = 0. \tag{3}$$

Spherical harmonics are normalized with respect to the natural integration measure (area) on the sphere, $\mathrm{d}\mu = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. They are also orthogonal:

$$\int_{S^2} \mathrm{d}\mu\, Y_j^{m*}(\theta, \phi) Y_{j'}^{m'}(\theta, \phi) = \delta_{jj'}\delta_{mm'}. \tag{4}$$

At this point it should be stressed that all these relations are valid as far as j is integer. Therefore, spherical harmonics provide just (true) representations for $SO(3)$. For j half-odd-integer other realizations, using spinors or holomorphic functions on the complex plane, are required.

The orthogonality relation (4) is more general than the usual orthogonality of a basis of functions $\left\{Y_j^m(\theta, \phi)\right\}$ for a given representation j, and this is due to the $\mathrm{e}^{\mathrm{i}m\phi}$ part of

[4] Associated Legendre functions $P_j^m(x)$ for integers j and m are also named associated Legendre polynomials since $P_j^m(\cos\theta)$ is a polynomial in $\cos\theta$ and $\sin\theta$.

the functions. It further expresses an orthogonality between arbitrary functions belonging to different representations (as happens for the unitary representations of finite groups, consequence of the Peter–Weyl theorem) and stating that the set of spherical harmonics is an orthonormal basis for the Hilbert space $L^2(S^2)$. This is provided by the following orthogonality relation of the associated Legendre functions:

$$\int_0^\pi P_j^m(\cos\theta) P_{j'}^m(\cos\theta)\sin\theta\, \mathrm{d}\theta = \frac{2}{2j+1}\frac{(j+m)!}{(j-m)!}\delta_{jj'}. \tag{5}$$

Note that the measure used for this orthogonality relation is the one obtained by reduction from the integration measure $\mathrm{d}\mu$ on the sphere.

Under conjugation, the spherical harmonics transform as

$$Y_j^{m*}(\theta,\phi) = (-1)^m Y_j^{-m}(\theta,\phi), \tag{6}$$

and this property is inherited from the following property of the associated Legendre functions (here it is crucial that j and m are integers):

$$P_j^{-m}(x) = (-1)^m \frac{(j-m)!}{(j+m)!} P_j^m(x), \qquad m = -j, -j+1, \ldots, j. \tag{7}$$

This last property implies that, for each integer j, only $j+1$, $\left\{P_j^m(x)\right\}_{m=-j}^{0}$, out of the $2j+1$ (well-defined) associated Legendre functions are linearly independent (note that equation (6) does not imply this dependence for spherical harmonics).

Therefore, the unitary representations of $SU(2)$ (or rather $SO(3)$) on the sphere in terms of spherical harmonics cannot be reduced to a representation on a half-circle, θ, $0 \leqslant \theta \leqslant \pi$, in terms of just the associated Legendre functions, since these are linearly dependent.

Another pathology of this realization on a half-circle is that, with the measure induced by the reduction, the associated Legendre functions are not orthogonal for fixed j and different values of m (equation (5) only implies that they are orthogonal for the same m and different values of j). Therefore, they definitely cannot carry a unitary irreducible representation of $SU(2)$ of dimension $2j+1$, since the operator J_3 would not be self-adjoint.

However, modifying the scalar product, they can be made orthogonal with respect to m:

$$\int_0^\pi P_j^m(\cos\theta) P_j^{m'}(\cos\theta)\frac{\mathrm{d}\theta}{\sin\theta} = \frac{1}{m}\frac{(j+m)!}{(j-m)!}\delta_{mm'}, \tag{8}$$

although at the price that the polynomial $P_j^0(\cos\theta)$ is not normalizable. This situation, nevertheless, is not better than the previous one, since now there are only j independent and normalizable functions. The measure $\frac{\mathrm{d}\theta}{\sin\theta}$ does not come from the reduction of the sphere to the half-circle, and is not invariant under the action of $SU(2)$. Also, the measure $\frac{\mathrm{d}\theta}{\sin\theta}$, which is the same (up to a sign) as the measure $\frac{\mathrm{d}x}{(1-x^2)}$, after the change of variables $x = \cos\theta$ is performed, rather than being associated with a compact carrier space, the half-circle $[0, \pi]$ or the closed interval $[-1, 1]$, it is associated with a non-compact carrier space, $(0, \pi)$ or $(-1, 1)$. In this way, only those associated Legendre functions which go to zero sufficiently fast at ± 1 are normalizable.

Thus, this framework does not seem to be very well suited to accommodate any of the unitary and irreducible representations of $SU(2)$. Rather, we shall see in section 4 that associated Legendre functions are best suited to describe certain finite-dimensional, non-unitary, representations of $SU(1,1)$.

Equation (3) admits more solutions than the ones specified up to now. In fact, for arbitrary complex numbers, j and m, it possess two kinds of solutions: the associated Legendre functions of the first and second kind, $P_j^m(x)$ and $Q_j^m(x)$, respectively. The set of the second kind solutions is always divergent at $x = \pm 1$, and therefore non-normalizable, for integers j and m, and that is the reason why they were not considered before.

For arbitrary j and m the relation (7) does not hold in general, and must be replaced by

$$P_j^{-m}(x) = \frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}\left[\cos(m\pi)P_j^m(x) - \frac{2}{\pi}\sin(m\pi)Q_j^m(x)\right]. \tag{9}$$

If m is integer, the expression (7) is recovered, but in any other case there is a mixture among functions of the first and second class.

Let us focus on the particular case of both j and m half-odd-integers (therefore $j \pm m$ are integers). Then the relation (9) gives

$$P_j^{-m}(x) = (-1)^{m+(1/2)}\frac{2}{\pi}\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}Q_j^m(x), \tag{10}$$

implying that first and second class solutions are the same, although now $P_j^m(x)$ and $P_j^{-m}(x)$ are linearly independent.

With respect to the measure $\mathrm{d}x$, which is the same (up to a sign) as the measure $\sin\theta\,\mathrm{d}\theta = \mathrm{d}(\cos\theta)$ after the change of variables $x = \cos\theta$ is performed, the associated Legendre functions $P_j^m(x)$ are normalizable for $m \leqslant \frac{1}{2}$, but non-normalizable for $m > \frac{1}{2}$. Furthermore, they are not orthogonal for the same j and different values of m, as happened for the integer case.

Therefore, although in the case j and m half-integer there are $2j+1$ linearly independent functions $\left\{P_j^m(x)\right\}_{m=-j}^{j}$, only $j+\frac{3}{2}$ are normalizable, $\left\{P_j^m(x)\right\}_{m=-j}^{1/2}$, but they are not orthogonal, so that they cannot carry a unitary irreducible representation of $SU(2)$ by themselves, not even when multiplying them by the phase factor $\mathrm{e}^{\mathrm{i}m\phi}$, since some of the states are non-normalizable. This is the reason why spherical harmonics are not well defined for j half-odd-integer.

With respect to the measure $\frac{\mathrm{d}x}{(1-x^2)}$ (or $\frac{\mathrm{d}\theta}{\sin\theta}$), only $j+\frac{1}{2}$ states are normalizable, $\left\{P_j^m(x)\right\}_{m=-j}^{-1/2}$, and these are orthogonal. We cannot conclude, however, that they carry a unitary irreducible representation of $SU(2)$, since there are exactly one half of the states, and the measure $\frac{\mathrm{d}\theta}{\sin\theta}$ is not invariant under $SU(2)$.

It should be stressed that similar conclusions can be drawn in the case in which j is an arbitrary positive number, and $m = -j, -j+1, \ldots$. In this case, all the associated Legendre functions $\left\{P_j^m(x)\right\}_{m=-j}^{\infty}$ are linearly independent, but only a few of them are normalizable. For the particular case of the measure $\frac{\mathrm{d}x}{(1-x^2)}$, only those with $m < 0$ are normalizable, and these are orthogonal, satisfying the orthogonality relations (8) with the factorials substituted by the appropriate Gamma functions. Thus, there are only $[j]+1$ independent and normalizable states.

Note that since $SU(2)$ is already simply connected, we cannot extend the range of allowed values for the index j of its representations, and therefore the space of independent and normalizable states obtained for j real cannot, by any means, support a representation of $SU(2)$.

3. Representations of $SU(2)$ on the plane: associated Laguerre polynomials

Let us review the construction of the representations of $SU(2)$ in polar coordinates in the plane in terms of two commuting boson operators a and b. This is the Schwinger representation of $SU(2)$, having the form

$$J_1 = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \qquad J_2 = \tfrac{1}{2}(a^\dagger b + b^\dagger a), \qquad J_3 = -\mathrm{i}\tfrac{1}{2}(a^\dagger b - b^\dagger a). \tag{11}$$

The total boson number operator $\hat{N} = a^\dagger a + b^\dagger b$ commutes with all $SU(2)$ generators, and is related to the Casimir J^2 by $J^2 = \frac{1}{4}\hat{N}(\hat{N}+2)$. Therefore, the spin index j of the $SU(2)$ representations is related to the total boson number N by $N = 2j$.

The boson operators can be realized in a two-dimensional harmonic oscillator space:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \qquad & a^\dagger &= \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right), \\ b &= \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right), \qquad & b^\dagger &= \frac{1}{\sqrt{2}}\left(y - \frac{\partial}{\partial y}\right), \end{aligned} \tag{12}$$

and performing the change to polar coordinates, $x = r\cos\phi$ and $y = r\sin\phi$, the total boson number operator and the third component of the $SU(2)$ generators read

$$\hat{N} = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + r^2\right) - 1, \qquad J_3 = -\frac{\mathrm{i}}{2}\frac{\partial}{\partial\phi}. \tag{13}$$

The common eigenvectors to both operators can be written as

$$\Psi_j^m(r,\phi) = \mathrm{e}^{-2\mathrm{i}m\phi}\, R_j^m(r) \tag{14}$$

where $R_j^m(r)$ satisfies the equation

$$\frac{1}{2}\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{4m^2}{r^2} + r^2\right)R_j^m(r) = (2j+1)R_j^m(r). \tag{15}$$

The solutions to this equation are

$$R_j^m(r) = \mathrm{e}^{-r^2/2}\, r^{-2m} L_{j+m}^{-2m}(r^2), \tag{16}$$

where $L_{j+m}^{-2m}(r^2)$ are the associated Laguerre polynomials. Using the change $\rho = r^2$ and defining $n = j+m$, the solutions are written as

$$R_j^n(\rho) = \mathrm{e}^{-\rho/2}\, y^{j-n} L_n^{2(j-n)}(\rho). \tag{17}$$

The integration measure is derived from the measure for the two oscillator system, $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\phi = (1/2)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$, and this is an invariant measure under $SU(2)$. With this measure, and for each j either integer or half-odd-integer, all the functions $\left\{\Psi_j^m(r,\phi)\right\}_{m=-j}^{j}$ are normalizable and orthogonal to each other. Therefore, in this case, j can take any positive integer or half-integer value, and m ranges from $-j, -j+1, \ldots, j-1, j$. Therefore, n takes the values $0, 1, \ldots, 2j$. Thus, all unitary and irreducible representations of $SU(2)$ of dimension $2j+1$ with j integer or half-odd-integer can be obtained in this way.

The normalized functions

$$\bar{\Psi}_j^m(r,\phi) = \sqrt{\frac{(j+m)!}{(j-m)!}}\,\Psi_j^m(r,\phi) \tag{18}$$

satisfy relations similar to those verified by spherical harmonics (but also valid for j half-odd-integer):

$$\bar{\Psi}_j^{m*}(r,\phi) = \bar{\Psi}_j^{-m}(r,\phi), \tag{19}$$

and these are derived from the corresponding properties of the associated Laguerre polynomials:

$$\rho^m L_{j-m}^{2m}(\rho) = \frac{(j+m)!}{(j-m)!}\rho^{-m} L_{j+m}^{-2m}(\rho). \tag{20}$$

Now we wonder whether a realization of the representations of $SU(2)$ in terms of just the variable r (or ρ) is possible or not. The answer is negative, as in the case of the sphere, and for the same reason. The orthogonality between different values of m of the complete wavefunctions $\Psi_j^m(r,\phi)$ is provided by the integration in the ϕ variable, in such a way that the functions $R_j^m(r)$ are not orthogonal with respect to the measure $r\,\mathrm{d}r$ (or $\mathrm{d}\rho$). This, along with the fact that some of the functions are linearly dependent according to equation (20), implies that the carrier space would consist of just $j+1$ for j integer or $j+\frac{1}{2}$ for j half-odd-integer, non-orthogonal (normalizable) independent states. And this set, of course, cannot support a unitary irreducible representation of $SU(2)$ since there are fewer states in the representation space than it should and the operator J_3 would not be self-adjoint.

Although the problem of the linear dependence of the states is unavoidable, the non-orthogonality of the eigenstates of J_3 can be circumvented by modifying the scalar product, as in the case of the associated Legendre functions. The new integration measure is in this case $\frac{\mathrm{d}r}{r}=\frac{1}{2}\frac{\mathrm{d}\rho}{\rho}$, and with respect to this scalar product the (linearly independent) functions $R_j^m(x)$ (or $R_j^n(\rho)$) are orthogonal to each other for different values of m:

$$\int_0^\infty R_j^m(x)R_j^{m'}\frac{\mathrm{d}r}{r}=\delta_{mm'}\frac{(j-m)!}{2|m|(j+m)!}. \tag{21}$$

Note that with this measure for j integer the state with $m=0$ is not normalizable.

Unlike in the case of the associated Legendre functions, for half-odd-integer j the states with $m>0$ are normalizable, but they are now linearly dependent (see equation (20)), and therefore the space of states has the same dimension as in the case of the associated Legendre functions.

Similar to the case of the realization in terms of the associated Legendre functions, we can extend the range of allowed values for j to any positive real number, obtaining the same results. Although the functions $R_j^m(r)$ are all linearly independent for $m=-j,-j+1,\ldots,$ only those with $m<0$ are normalizable with respect to the measure $\frac{\mathrm{d}r}{r}$. Therefore, there are $[j]+1$ states in the space of normalizable states, and these are orthogonal to each other for different values of m. In fact, expression (21) is valid in this case replacing the factorials with the appropriate Gamma functions.

Summarizing, when we try to represent the group $SU(2)$ in terms of a single variable, many difficulties emerge, such as the reduction of the number of independent functions and the non-orthogonality of eigenstates of the operator J_3. A possible solution is the use of a different scalar product which restores the orthogonality, but then some states become non-normalizable. This difficulties prevent such a realization, and suggest us that no realization in terms of a single variable exists for $SU(2)$.

Despite the differences between associated Legendre functions and associated Laguerre polynomials, the results presented here show that they have a very similar behaviour, and that, suitably choosing the scalar product in each case, a space of independent and normalizable functions can be found for the index j being integer, half-odd-integer or even real. In [1], one of these sets of functions, expressed in terms of the relativistic Hermite polynomials, which are related to the associated Legendre functions, was related to a non-unitary finite-dimensional representation of $SU(1,1)\approx SL(2,\mathbb{R})$. Using similar arguments, we can conclude that the sets of associated Laguerre polynomials here considered can also be related to non-unitary finite-dimensional representations of $SU(1,1)$.

4. The modified Pöschl–Teller potential

Let us apply the results of the previous sections to a pair of relevant quantum systems: the modified Pöschl–Teller potential and the Morse potential.

The modified Pöschl–Teller potential (MPT) is given by

$$V(x) = -\frac{D}{\cosh^2(\alpha x)}, \tag{22}$$

where $D > 0$ is the depth of the potential and α is related to its range.

The normalized solutions of the Schrödinger equation with this potential can be written as (see, for instance, [6])

$$\Psi_n^j(u) = N_n^j (1-u^2)^{\frac{j-n}{2}} C_n^{j-n+\frac{1}{2}}(u), \tag{23}$$

where $u = \tanh(\alpha x)$, $C_n^\alpha(u)$ are the Gegenbauer polynomials and the normalization constant is

$$N_n^j = \sqrt{\frac{\alpha n! \Gamma\left(j-n+\frac{1}{2}\right)}{\pi^{1/2}\Gamma(j-n)(2j-2n+1)_n}} \tag{24}$$

where $(a)_n$ is the Pochhammer symbol. The index j is related to the potential parameters by

$$j(j+1) = \frac{2mD}{\alpha^2\hbar^2}, \tag{25}$$

and n takes the values $n = 0, 1, \ldots, [j]$. It labels the energy levels in the form

$$E_n = -\frac{\hbar^2\alpha^2}{2m}(j-n)^2. \tag{26}$$

The scalar product used for the normalization of the states is given by the usual integration measure $\mathrm{d}x$, which renders the MPT Hamiltonian Hermitian. In terms of the variable $u = \tanh(\alpha x)$, the measure is given by $\mathrm{d}u/(1-u^2)$.

In [1], the Schrödinger equation, as well as its solutions and the integration measure, was derived from the Klein–Gordon-like equation associated with the discrete series representations of $SL(2,\mathbb{R})$ with negative Bargmann index $k = -j$. The solutions of this equation were expressed in terms of relativistic Hermite polynomials (RHP) with negative Bargmann index. These representations were identified with the finite-dimensional, non-unitary, irreducible representations of $SL(2,\mathbb{R})$. The non-unitarity of the representations manifested itself in that only $[j]+1$ out of the $2j+1$ states of the representation space were normalizable, and in that the ladder operators were not adjoint to each other. Restricting the representation to the subspace of the normalized, physically meaningful states, a set of non-local ladder operators, adjoint to each other, was obtained.

In [5], it was proved that through the change $u = \tanh(\alpha x)$, the Schrödinger equation for the MPT potential leads to the differential equation for the associated Legendre functions, equation (3). Therefore, the solutions could be written in terms of the associated Legendre functions. This is compatible with the solutions given in equation (23) since Gegenbauer polynomials and associated Legendre functions are related, and also it is compatible with the solutions given in [1] in terms of RHP with negative Bargmann index since they are also related up to a constant:

$$P_j^m(u) \approx (1-u^2)^{\frac{m}{2}} H_{j+m}^{-j}(u). \tag{27}$$

According to section 2, and since the measure derived from the physical measure $\mathrm{d}x$ is $\frac{\mathrm{d}u}{(1-u^2)}$ (see equation (8)), and not $\mathrm{d}u$, we can conclude that the space of solutions (23) does not support a unitary irreducible representation of $SU(2)$, refuting the general claim that this is the case [5]. Rather, as shown in [1], it carries a 'unitary', non-local realization of a finite-dimensional, non-unitary, non-local irreducible representation of $SU(1,1)$.

5. The Morse potential

The Morse potential has the form [3]

$$V(x) = D(\mathrm{e}^{-2\beta x} - 2\mathrm{e}^{-\beta x}), \tag{28}$$

where $D > 0$ is the depth of the potential and β is related to the range of the potential.

The normalized solutions of the Schrödinger equation associated with this potential can be written as (see, for instance, [3, 8])

$$\Psi_n^j(y) = N_n^j y^{j-n}\,\mathrm{e}^{-y/2}\,L_n^{2(j-n)}(y), \tag{29}$$

where $y = (2j+1)\,\mathrm{e}^{-\beta x}$, $L_n^\alpha(y)$ are associated Laguerre polynomials, and the normalization constant is

$$N_n^j = \sqrt{\frac{2\beta(j-n)\Gamma(n+1)}{\Gamma(2j-n+1)}}. \tag{30}$$

The index j is a positive number related to the potential parameters by

$$2j+1 = \frac{8mD}{\beta^2\hbar^2}, \tag{31}$$

and n takes the values $n = 0, 1, \ldots, [j]$. It labels the energy levels in the form

$$E_n = -\frac{\hbar^2\beta^2}{2m}(j-n)^2. \tag{32}$$

The scalar product used for the normalization of the states is again given by the usual integration measure dx, which renders the Morse Hamiltonian Hermitian. In terms of the variable $\rho = (2j+1)\,\mathrm{e}^{-\beta x}$, the measure is given by $\frac{\mathrm{d}\rho}{\rho}$.

In [5], it was proved that after the change of variables $r^2 = (2j+1)\,\mathrm{e}^{-\beta x}$ (note that $\rho = r^2$), the Schrödinger equation for the Morse potential transforms into the radial equation (15) of the $SU(2)$ representations on the plane.

According to section 3 and since the measure derived from the physical measure dx is $\frac{\mathrm{d}\rho}{\rho}$, and not $\frac{1}{2}\rho\,\mathrm{d}\rho$, we can conclude that the space of solutions (29) does not support a unitary irreducible representation of $SU(2)$, refuting the claim of [5]. Similar to the MPT case, it carries a 'unitary', non-local realization of a finite-dimensional, non-unitary representation of $SU(1,1)$, as will be shown elsewhere.

6. Conclusions

The main conclusion that we can extract from the previous discussion is that a quantum system, such as the modified Pöschl–Teller potential or the Morse potential, with a mixed spectrum consisting of a continuum and a finite number of bound states, should be described by a non-compact dynamical group (such as $SU(1,1) \approx SL(2,\mathbb{R})$) for both the continuum and the discrete spectrum. The continuum spectrum is given in terms of the principal continuous series of representations of $SU(1,1)$, whereas the discrete spectrum, with a finite number of bound states, is given in terms of a new class of 'unitary', non-local realizations of the finite-dimensional, non-unitary, representations of $SU(1,1)$.

It seems quite natural to conjecture that this result can be extrapolated to an arbitrary potential $V(x)$ with similar characteristics, i.e. with mixed spectrum consisting of a continuum and a finite number of bound states. This kind of potentials, if solvable, will be described by a non-compact dynamical group for both the continuum and the discrete spectra.

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